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Three loop $\overline{\rm MS}$ renormalization of QCD in the maximal abelian gauge

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Abstract. We determine the three loop anomalous dimensions of the quark, centre and off-diagonal gluons, centre and off-diagonal ghosts and the gauge fixing parameters in the maximal abelian gauge for an arbitrary colour group in the $\overline{\rm MS}$ renormalization scheme at three loops. We show that the three loop $\overline{\rm MS}$ β-function emerges from the renormalization of the centre gluon and also deduce the anomalous dimension of the BRST invariant dimension two mass operator. Moreover, we demonstrate that in the limit that the dimension of the centre of the group tends to zero, the anomalous dimensions of the quarks, off-diagonal gluons and off-diagonal ghosts tend to those of the quarks, gluons and ghosts of the Curci-Ferrari gauge respectively.

1 Introduction.

The multiloop renormalization of quantum chromodynamics (QCD), the quantum field theory underlying the strong interactions, has now been successfully determined at four loops in the $\overline{\text{MS}}$ scheme, [1, 2, 3, 4, 5, 6, 7]. Indeed the one loop β-function, [1], establishes the important property of asymptotic freedom. Further, with the need for more accurate theoretical results such as the precise way in which the coupling constant runs, higher loop corrections proved necessary. Subsequently, the scheme independent two loop result was computed in [2] prior to the three loop calculation of [4]. Given the large increase in the number of Feynman diagrams with loop order and the parallel problem of devising an algorithm to extract the divergence structure of difficult four loop master integrals, it was several years before the four loop β-function appeared, [6]. Indeed given the complexity of such a calculation, it was only technically possible with the intense use of the symbolic manipulation programme FORM, [8]. Though the three loop result of [4] also used computer technology and the MINCER algorithm, [9]. There was an underlying thread to all these computations which lay in a judicious choice of gauge in which to perform the calculation. Although the β-function is gauge independent, choosing a general covariant gauge, say, to carry out the calculations could have resulted in a large amount of extra unnecessary computation. This was avoided by considering the Feynman gauge where the gluon propagator reduces to one term proportional to a scalar field propagator. Only after the original Feynman gauge calculations were performed were computations with gluon propagators in the full covariant gauge subsequently carried out, [3, 5, 7]. These were necessary for other problems aside from justifying the full gauge parameter independence of the β-function.

For instance, the anomalous dimensions of the fields as functions of the covariant gauge parameter, α, were required for a variety of composite operator renormalizations such as those central to deep inelastic scattering. (See, for example, [10, 11].) Also, it has recently been established that there is an interesting relation, [12, 13], in respect of the dimension two BRST invariant operator which could play the role of a gluon mass. In [12, 13] it was demonstrated that in the Landau gauge its renormalization is not independent, being related to the gluon and ghost anomalous dimensions. This was observed by an explicit three loop computation in the $\overline{\text{MS}}$ scheme, [12]. More recently, the explicit renormalization has been determined at four loops through the provision of the Landau gauge gluon and ghost anomalous dimensions at that order, [14]. Significantly, similar identities for the analogous operator exist in other gauges such as the maximal abelian gauge (MAG), [15, 16], and in space-time dimensions other than four, [17]. Since these dimension two operators have been the subject of intense analytic investigation in various gauges in recent years, see, for instance [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and references therein, due to their condensation in a non-trivial vacuum, there is a clear need to renormalize QCD in this gauge. In particular the *explicit* values of all the anomalous dimensions are required as the first step in the extension of the local composite operator (LCO) method for QCD, [18], to the MAG in various colour groups. This would thus open up the possibility of extending the effective potential calculations in the Landau gauge, [18, 30], to situations beyond the few one loop $SU(2)$ MAG studies already considered, [19, 22, 26, 32, 33]. This is the main aim of this article where we will perform the full $\overline{\text{MS}}$ renormalization of QCD in the MAG for an arbitrary colour group to determine the explicit values of the anomalous dimensions with the renormalizability of the gauge having been discussed in [32, 34, 35, 36, 37, 38]. Though given the nature of the MAG construction where the colour group is split into its centre and off-diagonal sectors, we will make several assumptions about the group structure which we have checked are at least valid in $SU(2)$ and $SU(3)$. It is important to note that the only previous explicit renormalization of QCD in the MAG was at one loop and for the specific group $SU(2)$, [25, 32, 37, 38].

In referring to the MAG it is important to note at the outset that we are in fact considering the more general modified MAG as discussed in [22] for $SU(2)$. The reason for this is that the true MAG is defined in a similar fashion to the Landau gauge. However, by minimizing the square of the gauge potential over only the off-diagonal sector of the the colour group, as opposed to the full group in the usual covariant gauge situation, it transpires that the renormalization of the subsequent gauge fixed Lagrangian is singular. Therefore, analogously to the generalized Landau gauge or covariant gauge, a covariant gauge parameter, α, is introduced which is not to be confused with the parameter of the covariant gauges. With this non-zero α one has the modified MAG and as we will show, it is the renormalization of α itself which becomes singular as $\alpha \rightarrow 0$. However, all the remaining renormalization group functions are finite as $\alpha \rightarrow 0$ whence one obtains the true MAG anomalous dimensions. Moreover, as has been observed before, [19], the structure of the MAG renormalization has connections not only with the Landau gauge but also with the related non-linear covariant gauge known as the Curci-Ferrari gauge introduced in [39]. It will turn out that such connections will also prove useful for justifying our final three loop $\overline{\mathrm{MS}}$ anomalous dimensions.

Another motivation for considering the MAG rests in one of the original reasons why it was introduced. One possibility for the mechanism of confinement is the condensation of abelian monopoles which clearly originate from the centre of the colour group, [40, 41, 42]. In any calculations which seek to focus on this supposition, it makes sense to consider a gauge where the centre and off-diagonal fields are separately identified in the gauge fixing. Therefore, by establishing the renormalization structure at three loops in this gauge, one would expect the results will be useful, say, in any continuum matching one might have to do in lattice computations. On a final note we draw attention to another gauge in which QCD is renormalized and that is the background field gauge where the gauge field is split into a classical and quantum part, [43, 44, 45, 46]. The latter is regarded as the totally internal quantum fluctuation. In addition to the other three loop results referred to earlier, QCD has also been renormalized to the same order in this gauge, [45, 46, 47]. The main advantage of the background field gauge is the fact that the β-function emerges from the renormalization of the gluon field. In other words one needs only to consider a 2-point function rather than a 3-point function which considerably simplifies any explicit computation. Interestingly, the MAG, where the gluon field is split, but with respect to the colour property, has an analogous simplification which is that the centre gluon anomalous dimension is also equivalent to the β-function, [32]. This feature will be exploited here to reduce the number of Feynman diagrams we have to consider to perform the full three loop renormalization.

The paper is organized as follows. In section 2 we review how the MAG Lagrangian itself is constructed prior to summarizing the group theory results which were required for the three loop renormalization. This is a non-trivial exercise since the colour indices have to be identified either as originating in the centre of the Lie group or in the off-diagonal sector. The details of the full three loop renormalization are discussed in section 3 where the structure of the actual renormalization established with the algebraic renormalization formalism, [32], is reviewed. This section also contains the main results of the computation which is the determination of the explicit values of all renormalization group functions for the MAG. Finally, section 4 contains concluding remarks and the appendix contains the non-trivial Feynman rules used in the calculation.

2 Maximal abelian gauge.

We begin by recalling the essential features of the maximal abelian gauge fixing which depends on the parameter α. First, we note that the colour group generators are T^A where $1 \leq A \leq N_A$ and N_A is the dimension of the adjoint representation. Thus the group valued gauge field $\mathcal{A}_\mu$ can be decomposed as

$$\mathcal{A}_\mu ~=~ A^A_\mu T^A ~. \tag{2.1}$$

In considering the MAG the group generators are split into two sets. Those corresponding to the generators of the centre of the group, which themselves form a group, and the remaining set. For notational purposes we will use the indices i, j, k and l to denote centre elements and a, b, c and d to denote off-diagonal elements. Thus $\mathcal{A}_\mu$ can alternatively be decomposed as

$$\mathcal{A}_\mu ~=~ A^a_\mu T^a ~+~ A^i_\mu T^i \tag{2.2}$$

where we introduce the dimension of the centre by noting that $1 \leq i \leq N^d_A$ and allowing the off-diagonal indices to range over $1 \leq a \leq N^o_A$. Clearly

$$N^d_A ~+~ N^o_A ~=~ N_A \tag{2.3}$$

and, for instance, in the unitary groups $SU(N_c)$ we have $N^d_A = N_c - 1$ and $N^o_A = N_c(N_c-1)$. With this notation the QCD Lagrangian in general is, with the gauge fixing part L_{gf} to be specified,

$$L ~=~ -~ \frac{1}{4} G^A_{\mu\nu} G^{A\,\mu\nu} ~+~ i \bar{\psi} \not{D} \psi ~+~ L_{\mathrm{gf}} \tag{2.4}$$

where $G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f^{ABC} A^B_\mu A^C_\nu$, D_μ is the covariant derivative, there are N_f flavours of quarks, N_F is the dimension of the fundamental representation and g is the coupling constant. For the MAG the indices A are split into the two sectors giving

$$L ~=~ -~ \frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} ~-~ \frac{1}{4} G^i_{\mu\nu} G^{i\,\mu\nu} ~+~ i \bar{\psi} \not{D} \psi ~+~ L_{\mathrm{gf}} \tag{2.5}$$

where now L_{gf} is interpreted as the MAG gauge fixing term. This is constructed, see, for example, [22, 32], in the standard way by the BRST variation of a specific operator. In the usual covariant gauge fixing one uses

$$L^{\mathrm{cov}}_{\mathrm{gf}} ~=~ \delta \bar{\delta} \left[\tfrac{1}{2} A^A_\mu A^{A\,\mu} ~+~ \tfrac{1}{2} \alpha \bar{c}^A c^A \right] \tag{2.6}$$

where δ and $\bar{\delta}$ are the BRST and anti-BRST variations respectively, c^A is the ghost field and $\bar{c}^A$ is the anti-ghost field. In the MAG the gauge fixing term is chosen in a similar way. The off-diagonal sector is chosen as in the covariant gauge case but the diagonal sector is restricted to being in the Landau gauge to fully fix the gauge overall. It is not instructive to repeat all the additional technical details of the gauge fixing which have been discussed previously in [22, 32]. Therefore, for the MAG we take, [22, 32],

$$L_{\mathrm{gf}} ~=~ \delta \bar{\delta} \left[\tfrac{1}{2} A^a_\mu A^{a\,\mu} ~+~ \tfrac{1}{2} \alpha \bar{c}^a c^a ~+~ \tfrac{1}{2} \zeta A^i_\mu A^{i\,\mu} \right] ~+~ (1-\zeta) \delta \left[\bar{c}^i \partial^\mu A^i_\mu \right] \tag{2.7}$$

where the last term is included to ensure one can interpolate the results between the MAG and the Landau gauge according to how one chooses the additional parameter ζ. For instance, the Landau gauge corresponds to $\alpha = 0$ and $\zeta = 1$ and the (modified) MAG is $\alpha \neq 0$ but $\zeta = 0$. This particular gauge fixing was introduced in [22] and we have chosen to work with this version for various reasons. First, this Lagrangian has been examined from the algebraic renormalization point of view and the Slavnov-Taylor identities have been established. Second,

and more crucially for the current article, in a computation of the magnitude of the three loop MAG renormalization it is important to recognise that calculating with an arbitrary α and ζ allows us to check the correctness of, say, our programming and resultant renormalization constants. In particular the Landau gauge three loop anomalous dimensions ought to correctly emerge from the computation prior to specifying the MAG values of the parameter ζ. This is actually a non-trivial point since we have to perform the group theory manipulations for the split group and not the full group as one would do in an ordinary covariant gauge fixed Lagrangian where the Casimir structure resulting from group identities is already well established.

With the MAG gauge fixing, (2.6), it is elementary to perform the BRST and anti-BRST variations, which are given by

$$\begin{aligned}
\delta A^a_\mu &= - \left(\partial_\mu c^a + g f^{ajc} A^j_\mu c^c + g f^{abc} A^b_\mu c^c + g f^{abk} A^b_\mu c^k \right) \\
\delta c^a &= g f^{abk} c^b c^k + \frac{1}{2} f^{abc} c^b c^c \quad , \quad \delta \bar{c}^a = b^a \quad , \quad \delta b^a = 0 \, , \\
\delta A^i_\mu &= - \left(\partial_\mu c^i + g f^{ibc} A^b_\mu c^c \right) \quad , \quad \delta c^i = \frac{1}{2} g f^{ibc} c^b c^c \quad , \quad \delta \bar{c}^i = b^i \quad , \quad \delta b^i = 0
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\bar{\delta} A^a_\mu &= - \left(\partial_\mu c^a + g f^{ajc} A^j_\mu c^c + g f^{abc} A^b_\mu c^c + g f^{abk} A^b_\mu c^k \right) \\
\bar{\delta} c^a &= - b^a + g f^{abc} c^b \bar{c}^c + g f^{abk} c^b \bar{c}^k + g f^{abk} \bar{c}^b c^k \\
\bar{\delta} \bar{c}^a &= g f^{abk} \bar{c}^b \bar{c}^k + \frac{1}{2} g f^{abc} \bar{c}^b \bar{c}^c \quad , \quad \bar{\delta} b^a = - g f^{abc} b^b \bar{c}^c - g f^{abk} b^b \bar{c}^k + g f^{abk} \bar{c}^b b^k \, , \\
\bar{\delta} A^i_\mu &= - \left(\partial_\mu \bar{c}^i + g f^{ibc} A^b_\mu \bar{c}^c \right) \quad , \quad \bar{\delta} c^i = - b^i + g f^{ibc} c^b \bar{c}^c \quad , \quad \bar{\delta} \bar{c}^i = \frac{1}{2} g f^{ibc} \bar{c}^b \bar{c}^c \, , \\
\bar{\delta} b^i &= - g f^{ibc} b^b \bar{c}^c
\end{aligned} \tag{2.9}$$

respectively, to obtain the gauge fixed MAG Lagrangian

$$\begin{aligned}
L_{\mathrm{gf}} &= \frac{\alpha}{2} b^a b^a + b^a \partial^\mu A^a_\mu + \frac{\bar{\alpha}}{2} b^i b^i + b^i \partial^\mu A^i_\mu + \bar{c}^a \partial^\mu \partial_\mu c^a + \bar{c}^i \partial^\mu \partial_\mu c^i \\
&- g \, b^a \left[(1-\zeta) f^{abk} A^b_\mu A^{k\,\mu} - \frac{1}{2} \alpha f^{abc} \bar{c}^b c^c - \alpha f^{abk} \bar{c}^b c^k \right] \\
&+ g \left[(1-\zeta) f^{abk} A^a_\mu \bar{c}^k \partial^\mu c^b - \zeta f^{abk} A^a_\mu \partial^\mu c^b \bar{c}^k - f^{abc} A^a_\mu \partial^\mu c^b \bar{c}^c - \zeta f^{abk} A^a_\mu c^b \partial^\mu \bar{c}^k \right. \\
&\quad - f^{abk} \partial^\mu A^a_\mu c^b \bar{c}^k - f^{abc} \partial^\mu A^a_\mu \bar{c}^b c^c - f^{abk} \partial^\mu A^a_\mu \bar{c}^b c^k - (2-\zeta) f^{abk} A^k_\mu \bar{c}^a \partial^\mu \bar{c}^b \\
&\quad \left. - f^{abk} \partial^\mu A^k_\mu \bar{c}^a c^b \right] \\
&+ g^2 \left[(1-\zeta) f^{acbd}_d A^a_\mu A^{b\,\mu} \bar{c}^c c^d + (1-\zeta) f^{adcj}_o A^a_\mu A^{j\,\mu} \bar{c}^c c^d + (1-\zeta) f^{alcj}_o A^a_\mu A^{j\,\mu} \bar{c}^c c^l \right. \\
&\qquad \left. + (1-\zeta) f^{cjdi}_o A^i_\mu A^{j\,\mu} \bar{c}^c c^d - \frac{\alpha}{4} f^{abcd}_d \bar{c}^a \bar{c}^b c^c c^d - \frac{\alpha}{8} f^{abcd}_o \bar{c}^a \bar{c}^b c^c c^d - \frac{\alpha}{4} f^{abcl}_o \bar{c}^a \bar{c}^b c^c c^l \right]
\end{aligned} \tag{2.10}$$

where we have introduced the term $\frac{1}{2} \bar{\alpha} b^i b^i$ to fix the residual gauge freedom in the full gauge field, [32]. We have also simplified the notation by defining

$$f^{ABCD}_d = f^{iAB} f^{iCD} \quad , \quad f^{ABCD}_o = f^{eAB} f^{eCD} \tag{2.11}$$

for the 4-point vertices. For perturbative computations the auxiliary fields b^a and b^i are eliminated by their equations of motion

$$\begin{aligned}
b^a &= - \frac{1}{\alpha} \left[\partial^\mu A^a_\mu - (1-\zeta) g f^{abk} A^b_\mu A^{k\,\mu} - \frac{1}{2} \alpha g f^{abc} \bar{c}^b c^c - \alpha g f^{abk} \bar{c}^b c^k \right] \\
b^i &= - \frac{1}{\bar{\alpha}} \partial^\mu A^i_\mu
\end{aligned} \tag{2.12}$$

to obtain the MAG Lagrangian in the form we will renormalize it, [32],

$$
\begin{aligned}
L_{\mathrm{gf}} &= -\frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \frac{1}{2\bar{\alpha}}\left(\partial^\mu A^i_\mu\right)^2 + \bar{c}^a\partial^\mu\partial_\mu c^a + \bar{c}^i\partial^\mu\partial_\mu c^i \\
&+ g\left[(1-\zeta)f^{abk}A^a_\mu\bar{c}^k\partial^\mu c^b - \zeta f^{abk}A^a_\mu\partial^\mu c^b\bar{c}^k - f^{abc}A^a_\mu\bar{c}^b\partial^\mu c^c - \zeta f^{abk}A^a_\mu\bar{c}^b\partial^\mu c^k\right. \\
&\quad - \frac{(1-\zeta)}{\alpha}f^{abk}\partial^\mu A^a_\mu A^b_\nu A^{k\,\nu} - f^{abk}\partial^\mu A^a_\mu c^b\bar{c}^k - \frac{1}{2}f^{abc}\partial^\mu A^a_\mu\bar{c}^b c^c \\
&\quad \left. - (2-\zeta)f^{abk}A^k_\mu\bar{c}^a\partial^\mu\bar{c}^b - f^{abk}\partial^\mu A^k_\mu\bar{c}^b c^c\right] \\
&+ g^2\left[(1-\zeta)f^{acbd}_d A^a_\mu A^{b\,\mu}\bar{c}^c c^d - \frac{(1-\zeta)^2}{2\alpha}f^{akbl}_o A^a_\mu A^{b\,\mu}A^k_\nu A^{l\,\nu} + (1-\zeta)f^{adcj}_o A^a_\mu A^{j\,\mu}\bar{c}^c c^d\right. \\
&\quad - \frac{(1-\zeta)}{2}f^{ajcd}_o A^a_\mu A^{j\,\mu}\bar{c}^c c^d + (1-\zeta)f^{ajcl}_o A^a_\mu A^{j\,\mu}\bar{c}^c c^l + (1-\zeta)f^{alcj}_o A^a_\mu A^{j\,\mu}\bar{c}^c c^l \\
&\quad - (1-\zeta)f^{cjdi}_o A^i_\mu A^{j\,\mu}\bar{c}^c c^d - \frac{\alpha}{4}f^{abcd}_d\bar{c}^a\bar{c}^b c^c c^d - \frac{\alpha}{8}f^{abcd}_o\bar{c}^a\bar{c}^b c^c c^d + \frac{\alpha}{8}f^{acbd}_o\bar{c}^a\bar{c}^b c^c c^d \\
&\quad \left. - \frac{\alpha}{4}f^{abcl}_o\bar{c}^a\bar{c}^b c^c c^l + \frac{\alpha}{4}f^{acbl}_o\bar{c}^a\bar{c}^b c^c c^l - \frac{\alpha}{4}f^{albc}_o\bar{c}^a\bar{c}^b c^c c^l + \frac{\alpha}{2}f^{akbl}_o\bar{c}^a\bar{c}^b c^k c^l\right] . \qquad (2.13)
\end{aligned}
$$

where it is understood that the parameter $\bar{\alpha}$, which is distinct from α, is set to zero after our renormalization. Having constructed the full MAG Lagrangian as a function of the parameters α, ζ and $\bar{\alpha}$ we note that the full set of non-zero Feynman rules generated from (2.13) are given in appendix A.

Since we will be performing our calculation for an arbitrary colour group but with the group algebra split into centre and off-diagonal sectors, we close this section by discussing the main properties of the Lie algebra which were required. To construct the necessary lemmas we recall that the Lie algebra and basic Casimirs for the full group as well as the Jacobi identity are

$$
\begin{aligned}
\left[T^A,T^B\right] &= if^{ABC}T^C \\
f^{ACD}f^{BCD} &= C_A\delta^{AB} ~~,~~ T^AT^A ~=~ C_F I ~~,~~ \mathrm{Tr}\left(T^AT^B\right) ~=~ T_F\delta^{AB} \\
0 &= f^{ABE}f^{CDE} ~+~ f^{BCE}f^{ADE} ~+~ f^{CAE}f^{BDE} ~. \qquad (2.14)
\end{aligned}
$$

From the Lie algebra we have that $f^{ijk}=0$ and $f^{ija}=0$ which enshrines the centre property in the algebraic manipulations. So the second equation of (2.14) gives the relations

$$
\begin{aligned}
C_A\delta^{ab} &= f^{acd}f^{bcd} ~+~ 2f^{acj}f^{bcj} \\
C_A\delta^{ij} &= f^{icd}f^{jcd} ~. \qquad (2.15)
\end{aligned}
$$

To proceed we make the assumption that $f^{acd}f^{bcd}$ is proportional to δ^{ab} which is certainly true for $SU(2)$ and we have checked it is also valid in $SU(3)$. In the group theory discussion which follows, it is important to bear in mind that for groups where this simplifying feature is not present then one would have to proceed with $f^{acd}f^{bcd}$ being proportional to a symmetric rank two tensor. Taking contractions of (2.15) leads to

$$
f^{iab}f^{iab} ~=~ N^d_A C_A ~~,~~ f^{abc}f^{abc} ~=~ \left[N^o_A - 2N^d_A\right]C_A \qquad (2.16)
$$

where N^d_A is the dimension of the centre and N^o_A is the dimension of the complement of the centre. Hence,

$$
f^{icd}f^{jcd} ~=~ C_A\delta^{ij} ~~,~~ f^{acj}f^{bcj} ~=~ \frac{N^d_A}{N^o_A}C_A\delta^{ab} ~~,~~ f^{acd}f^{bcd} ~=~ \frac{[N^o_A - 2N^d_A]}{N^o_A}C_A\delta^{ab} ~. \qquad (2.17)
$$

With these elementary results we can use the Jacobi identity to establish several useful relations which were used extensively throughout the computation

$$\begin{aligned} f^{apq} f^{bpr} f^{cqr} &= \frac{[N_A^o - 3N_A^d]}{2N_A^o} C_A f^{abc} \quad , \quad f^{apq} f^{bpi} f^{cqi} = \frac{N_A^d}{2N_A^o} C_A f^{abc} \\ f^{ipq} f^{bpr} f^{cqr} &= \frac{[N_A^o - 2N_A^d]}{2N_A^o} C_A f^{ibc} \quad , \quad f^{ipq} f^{bpj} f^{cqj} = \frac{N_A^d}{N_A^o} C_A f^{ibc} \, . \end{aligned} \tag{2.18}$$

For the group generators, we have

$$\mathrm{Tr} \left(T^a T^b \right) = T_F \delta^{ab} \quad , \quad \mathrm{Tr} \left(T^a T^i \right) = 0 \quad , \quad \mathrm{Tr} \left(T^i T^j \right) = T_F \delta^{ij} \tag{2.19}$$

and we make the assumption that $T^i T^i$ is proportional to the unit matrix which is certainly true for the groups $SU(N)$. Then, from (2.14) we have

$$T^i T^i = \frac{T_F}{N_F} N_A^d I \tag{2.20}$$

after contracting $\mathrm{Tr} \left(T^i T^j \right)$. Hence,

$$T^a T^a = \left[C_F - \frac{T_F}{N_F} N_A^d \right] I \, . \tag{2.21}$$

It therefore follows from the Lie algebra itself that

$$\begin{aligned} T^b T^a T^b &= \left[C_F - \frac{C_A}{2} - \frac{T_F}{N_F} N_A^d + \frac{C_A N_A^d}{2N_A^o} \right] T^a \quad , \quad T^i T^a T^i = \left[\frac{T_F}{N_F} N_A^d - \frac{C_A N_A^d}{2N_A^o} \right] T^a \\ T^a T^i T^a &= \left[\frac{T_F}{N_F} N_A^o - \frac{C_A}{2} \right] T^i \quad , \quad T^j T^i T^j = \frac{T_F}{N_F} N_A^d T^i \, . \end{aligned} \tag{2.22}$$

As a consistency check on these results adding the first pair together recovers the usual result

$$T^B T^A T^B = \left[C_F - \frac{C_A}{2} \right] T^A \tag{2.23}$$

for a free off-diagonal index. Summing the final pair is also consistent with this result after use of the relation

$$C_F N_F = \left[N_A^o + N_A^d \right] T_F \tag{2.24}$$

which follows from taking the trace of $T^A T^A$. Next, given the Lie algebra it is straightforward to construct the useful lemmas

$$\begin{aligned} f^{abc} T^b T^c &= \frac{i[N_A^o - 2N_A^d]}{2N_A^o} C_A T^a \quad , \quad f^{abj} T^b T^j = \frac{iN_A^d}{2N_A^o} C_A T^a \\ f^{ibc} T^b T^c &= \frac{i}{2} C_A T^i \, . \end{aligned} \tag{2.25}$$

Whilst these results proved to be the workhorse for the full three loop computation as well, it turned out that at three loops it was quicker to include additional lemmas to speed up the group theory computation of our FORM programmes. These were derived from several applications of the Jacobi identities and are

$$\begin{aligned} f^{apq} f^{brs} f^{qms} f^{cmt} f^{prt} &= 0 \\ f^{apq} f^{bjs} f^{qms} f^{cmt} f^{pjt} &= 0 \end{aligned}$$

$$
\begin{aligned}
f^{apq} f^{brs} f^{qjs} f^{cjt} f^{prt} &= 0 \\
f^{apq} f^{brj} f^{qmj} f^{cmk} f^{prk} &= \frac{N^{d^2}_A C^2_A}{4 N^{o2}_A} f^{abc} \\
f^{apj} f^{brs} f^{jms} f^{imt} f^{prt} &= \frac{N^d_A [N^o_A - 2 N^d_A] C^2_A}{4 N^{o2}_A} f^{abi} \\
f^{apj} f^{bks} f^{jms} f^{imt} f^{pkt} &= \frac{N^{d^2}_A C^2_A}{N^{o2}_A} f^{abi} \\
f^{apq} f^{brs} f^{qms} f^{imt} f^{prt} &= 0 \\
f^{apj} f^{bks} f^{jms} f^{cmt} f^{pkt} &= \frac{N^{d^2}_A C^2_A}{4 N^{o2}_A} f^{abc}
\end{aligned}
\tag{2.26}
$$

where we note that the indices m, p, q, r, s and t are also regarded as off-diagonal. Given the structure of the three loop Green's functions these relations were sufficient for handling the group theory associated with the gluon 2-point functions. In that case for any Feynman diagram one has at most six structure functions contracted together with two free external group indices. However, in the renormalization of the $A^a_\mu \bar{c}^i c^b$ vertex at most seven structure functions are contracted together with three free group indices. For this case the Green's function was multiplied by an additional structure function to leave a scalar group string to be simplified. The route to achieving this, aside from applying the rules discussed so far, was to follow the approach used for the structure constants of the full group in that in that case they correspond to the group generators in the adjoint representation. In other words one replaces the structure constants by

$$
\left(T^A_{\mathrm{adj}} \right)_{BC} ~=~ -~ i f^{ABC}
\tag{2.27}
$$

and then applies the usual Lie algebra properties to T^A_{adj} with the proviso that one evaluates identities in the adjoint representation. For instance, the result

$$
T^B T^C T^A T^B T^C ~=~ \left(C_F - C_A \right) \left(C_F - \tfrac{1}{2} C_A \right) T^A
\tag{2.28}
$$

implies

$$
T^B_{\mathrm{adj}} T^C_{\mathrm{adj}} T^A_{\mathrm{adj}} T^B_{\mathrm{adj}} T^C_{\mathrm{adj}} ~=~ 0 ~.
\tag{2.29}
$$

For the MAG calculation one can also use this strategy provided one appreciates that the structure constants with two or more centre indices are identically zero which means that the non-trivial structure constants have at least two off-diagonal indices. These are therefore regarded as the matrix indices which leaves the third index as the free generator index and this can either be centre or off-diagonal. More significantly one must be careful in regarding these objects as nothing more than matrices and not as representations of the group generators since the matrix indices are only elements of the off-diagonal sector which is not closed in the group sense. In light of this to differentiate from the adjoint representation of the full group we therefore choose to define the analogous object S^A by

$$
\left(S^a \right)_{bc} ~=~ f^{abc} ~~,~~ \left(S^i \right)_{bc} ~=~ f^{ibc}
\tag{2.30}
$$

where the two matrix indices will always be off-diagonal. Subsequently, given all the previous lemmas it only remains to resolve objects of the form

$$
\mathrm{tr} \left(S^A S^B S^C S^D \right) \mathrm{tr} \left(S^A S^B S^C S^D \right)
\tag{2.31}
$$

and

$$
\mathrm{tr} \left(S^A S^B S^C S^D S^A S^B S^C S^D \right)
\tag{2.32}
$$

where the trace is the usual matrix trace and we use tr in contradistinction to the Tr of the full group. Such structures are known to occur in higher loop calculations in QCD when S^A is formally replaced by T^A, [6], but not until four loops where they arise only in terms involving the simple pole in ϵ where $d = 4 - 2\epsilon$. Therefore, to ensure renormalizability they either have to vanish or cancel since at three loops they can potentially occur in the double and triple poles in ϵ as well as the simple one. As these two structures emerge with the summed indices in various combinations of centre and off-diagonal indices, it is appropriate to relate them to a common term via the relations

$$
\begin{aligned}
\mathrm{tr}\left(S^iS^jS^kS^l\right)\mathrm{tr}\left(S^iS^jS^kS^l\right) &= -\,\mathrm{tr}\left(S^iS^jS^kS^d\right)\mathrm{tr}\left(S^iS^jS^kS^d\right) + \left[6N^d_A + N^o_A\right]\frac{N^{d^3}_A C^4_A}{4N^{o3}_A} \\
\mathrm{tr}\left(S^iS^jS^kS^d\right)\mathrm{tr}\left(S^iS^jS^kS^d\right) &= -\,\mathrm{tr}\left(S^iS^jS^cS^d\right)\mathrm{tr}\left(S^iS^jS^cS^d\right) \\
&\quad - \left[4N^{d^2}_A - N^{o2}_A\right]\frac{N^{d^2}_A C^4_A}{8N^{o3}_A} \\
\mathrm{tr}\left(S^iS^jS^cS^d\right)\mathrm{tr}\left(S^iS^jS^cS^d\right) &= -\,\mathrm{tr}\left(S^iS^bS^cS^d\right)\mathrm{tr}\left(S^iS^bS^cS^d\right) \\
&\quad + \mathrm{tr}\left(S^iS^bS^cS^dS^iS^bS^cS^d\right) + \mathrm{tr}\left(S^iS^jS^cS^dS^iS^jS^cS^d\right) \\
&\quad - \left[5N^{d^2}_A - 4N^d_A N^o_A + N^{o2}_A\right]\frac{N^d_A C^4_A}{8N^{o3}_A}
\end{aligned}
\tag{2.33}
$$

and

$$
\mathrm{tr}\left(S^iS^jS^cS^dS^iS^jS^cS^d\right) = 0 \tag{2.34}
$$

which are readily established by use of the Lie algebra and the Jacobi identity. In the actual renormalization of the $A^a_\mu \bar{c}^i c^b$ vertex this leaves the two as yet unevaluated structures as $\mathrm{tr}\left(S^iS^bS^cS^d\right)\mathrm{tr}\left(S^iS^bS^cS^d\right)$ and $\mathrm{tr}\left(S^iS^bS^cS^dS^iS^bS^cS^d\right)$. It turns out that when the pole parts of all the Feynman diagrams for this renormalization are added up then the coefficients of these structures is finite.

3 Renormalization.

Having derived the MAG Lagrangian we now turn to the details of its renormalization. First, in renormalizing a renormalizable quantum field theory one ordinarily introduces renormalization constants for all the fields and parameters in the Lagrangian. For field theories possessing symmetries such as a gauge symmetry these renormalization constants are not necessarily all independent. The underlying symmetry can constain several or more to be related. To determine such relations, one can apply techniques such as algebraic renormalization, [48], which ensures the Lagrangian is stable under quantum corrections. In [32] this approach has been applied to (2.13) and several interesting relations emerge. For instance, it turns out that the anomalous dimension of the centre gluons is proportional to the QCD β-function. This is a useful result since for this gauge fixed version of QCD it means that one does not have to renormalize a 3-point vertex to determine the known three loop β-function of [4]. Instead one needs only to consider the centre gluon 2-point function. From a practical computational point of view this is a significant observation which we exploit later. Moreover, a similar property is also present in the background field gauge where the anomalous dimension of the background gluon is simply related to the coupling constant renormalization, [45, 46]. Although we are a priori aware of the relation of the centre gluon renormalization to that of the coupling constant renormalization in the MAG, in defining our renormalization constants we choose at the outset to leave this result

to emerge in the computation rather than put restrictions on the initial setup. Therefore, we define the renormalization constants as

$$
\begin{aligned}
A^{a\,\mu}_{\mathrm{o}} &= \sqrt{Z_A}\, A^{a\,\mu} \quad , \quad A^{i\,\mu}_{\mathrm{o}} ~=~ \sqrt{Z_{A^i}}\, A^{i\,\mu} \quad , \quad c^a_{\mathrm{o}} ~=~ \sqrt{Z_c}\, c^a \quad , \quad \bar{c}^a_{\mathrm{o}} ~=~ \sqrt{Z_c}\, \bar{c}^a ~, \\
c^i_{\mathrm{o}} &= \sqrt{Z_{c^i}}\, c^i \quad , \quad \bar{c}^i_{\mathrm{o}} ~=~ \frac{\bar{c}^i}{\sqrt{Z_{c^i}}} \quad , \quad \psi_{\mathrm{o}} ~=~ \sqrt{Z_\psi}\psi ~, \\
g_{\mathrm{o}} &= \mu^\epsilon Z_g\, g \quad , \quad \alpha_{\mathrm{o}} ~=~ Z^{-1}_\alpha Z_A\, \alpha \quad , \quad \bar{\alpha}_{\mathrm{o}} ~=~ Z^{-1}_{\alpha^i} Z_{A^i}\, \bar{\alpha} \quad , \quad \zeta_{\mathrm{o}} ~=~ Z_\zeta \zeta
\end{aligned}
\tag{3.1}
$$

where μ is the renormalization scale introduced to ensure the coupling constant is dimensionless in d dimensions, the subscript $_{\mathrm{o}}$ denotes the bare quantity and the subscript i in the field subscripts of the renormalization constants is included to indicate that they correspond to centre objects and there is clearly no summation over repeated indices in this instance. In writing down (3.1) from [32] we have chosen, by contrast to A^i_μ, to encode the structure of the centre ghost renormalization. In particular the anti-centre ghost and centre ghost renormalizations are, contrary to the usual covariant gauge ghost renormalization, inverses of each other and not equal. This property emerges from the algebraic renormalization analysis, [32]. From a practical point of view this means that the centre ghost 2-point function cannot be used to determine Z_{c^i} since it would be finite, [32]. Instead to find Z_{c^i} one has to renormalize the 3-point $A^a_\mu \bar{c}^i c^b$ vertex once the coupling constant and off-diagonal gluon and off-diagonal ghost wave function renormalizations have been determined at that particular loop order. Therefore, in the MAG one has still at least one 3-point function renormalization to perform.

However, the benefit in determining Z_{c^i} rests in the fact that the dimension two BRST invariant operator

$$
\mathcal{O} ~=~ \tfrac{1}{2} A^a_\mu A^{a\,\mu} ~+~ \alpha \bar{c}^a c^a
\tag{3.2}
$$

possesses an interesting renormalization structure, [15, 32]. It transpires that its anomalous dimension is not independent but satisfies

$$
\gamma_{\mathcal{O}}(a) ~=~ -~ \frac{\beta(a)}{a} ~+~ \gamma_{c^i}(a)
\tag{3.3}
$$

where

$$
a ~=~ \frac{g^2}{16\pi^2}
\tag{3.4}
$$

and for completeness its associated renormalization constant is defined as

$$
\mathcal{O}_{\mathrm{o}} ~=~ Z_{\mathcal{O}}\, \mathcal{O} ~.
\tag{3.5}
$$

Therefore, it will be straightforward to deduce $\gamma_{\mathcal{O}}(a)$ from explicit knowledge of $\gamma_{c^i}(a)$. This is one of the key results required for a two loop extension of the LCO method to the condensation of $\mathcal{O}$ in the MAG and will be one of the main results of the article. That such a relation is present in the MAG is not specific to this gauge. A similar relation exists in the Landau gauge, [12, 13], for the analogous operator where the indices range over the full colour group. We have also introduced a renormalization constant in (3.1) for the interpolating parameter ζ. However, since we are only interested in the renormalization of the MAG itself which corresponds to the fixed point value of $\zeta = 0$ it turns out that for the MAG renormalization the explicit form of Z_ζ is not required since it will always be multiplied by zero and there are no singularities in ζ in the Feynman rules.

We now turn to the technical details of the renormalization of (2.13) at three loops in the $\overline{\mathrm{MS}}$ scheme. First, the renormalization group functions we will determine are deduced from the

Green's function	One loop	Two loop	Three loop	Total
$A^a_\mu\, A^b_\nu$	6	131	6590	6727
$A^i_\mu\, A^j_\nu$	3	54	2527	2584
$c^a\, \bar{c}^b$	3	81	4006	4090
$\psi\, \bar{\psi}$	2	27	979	1008
$A^a_\mu\, \bar{c}^i\, c^b$	5	287	22621	22913
Total	19	580	36723	37322

Table 1. Number of Feynman diagrams for each Green's function for the MAG renormalization.

explicit respective renormalization constants themselves for the MAG, via

$$\begin{aligned}
\gamma_A(a) &= \beta(a)\frac{\partial}{\partial a}\ln Z_A \;+\; \alpha\gamma_\alpha(a)\frac{\partial}{\partial \alpha}\ln Z_A \\
\gamma_\alpha(a) &= \left[\beta(a)\frac{\partial}{\partial a}\ln Z_\alpha \;-\; \gamma_A(a)\right]\left[1 \;-\; \alpha\frac{\partial}{\partial \alpha}\ln Z_\alpha\right]^{-1} \\
\gamma_{A^i}(a) &= \beta(a)\frac{\partial}{\partial a}\ln Z_{A^i} \;+\; \alpha\gamma_\alpha(a)\frac{\partial}{\partial \alpha}\ln Z_{A^i} \\
\gamma_{\alpha^i}(a) &= \beta(a)\frac{\partial}{\partial a}\ln Z_{\alpha^i} \;+\; \alpha\gamma_\alpha(a)\frac{\partial}{\partial \alpha}\ln Z_{\alpha^i} - \gamma_{A^i}(a) \\
\gamma_c(a) &= \beta(a)\frac{\partial}{\partial a}\ln Z_c \;+\; \alpha\gamma_\alpha(a)\frac{\partial}{\partial \alpha}\ln Z_c \\
\gamma_{c^i}(a) &= \beta(a)\frac{\partial}{\partial a}\ln Z_{c^i} \;+\; \alpha\gamma_\alpha(a)\frac{\partial}{\partial \alpha}\ln Z_{c^i} \\
\gamma_\psi(a) &= \beta(a)\frac{\partial}{\partial a}\ln Z_\psi \;+\; \alpha\gamma_\alpha(a)\frac{\partial}{\partial \alpha}\ln Z_\psi \\
\gamma_{\mathcal{O}}(a) &= \beta(a)\frac{\partial}{\partial a}\ln Z_{\mathcal{O}} \;+\; \alpha\gamma_\alpha(a)\frac{\partial}{\partial \alpha}\ln Z_{\mathcal{O}}
\end{aligned} \tag{3.6}$$

where, similar to the Curci-Ferrari gauge, we have not assumed that $Z_\alpha = 1$. Though we have set $\bar{\alpha} = 0$ and $\zeta = 0$. Since these are the renormalization group functions we require, we will therefore renormalize the centre and off-diagonal gluon 2-point function, the off-diagonal ghost 2-point function, the quark 2-point function and the $A^a_\mu \bar{c}^i c^b$ 3-point function all at three loops. For these Green's functions the Feynman diagrams were generated with the QGRAF package, [49], and the specific number of diagrams at each loop order and Green's function are summarized in Table 1. By contrast, to indicate the magnitude of the MAG renormalization using the Feynman rules of the appendix, we have provided a similar diagram count for the Curci-Ferrari gauge three loop renormalization of [12] in Table 2. To proceed we convert the QGRAF output format to the electronic notation used by the MINCER algorithm, [9], as written in the symbolic manipulation package FORM, [8, 50], in terms of diagram topology and internal momentum routing. The MINCER algorithm is then applied to all 37322 Feynman diagrams required for the full renormalization. Though it ought to be noted that given that the main group theory is carried out prior to determining the divergence structure of a diagram the value of a graph could be zero purely from group considerations. For instance, when one has a one loop gluonic self-energy subgraph anywhere where one external leg of the subgraph is in the centre of the group and the other in the off-diagonal sector, then that graph is trivially zero since $f^{icd} f^{bcd} = 0$. To appreciate the benefit of the centre gluon anomalous dimension relation to the β-function, if such a relation did not exist then one would have to compute a 3-point function in addition to the ones listed in Table 1. The easiest one from a computer algebra point of view is the quark gluon vertex. For an off-diagonal gluon the figures for the corresponding

first three columns of Table 1 are 5, 217 and 13108, and 3, 137 and 8150 for a centre gluon quark vertex.

Green's function	One loop	Two loop	Three loop	Total
$A^A_\mu \, A^B_\nu$	3	19	282	304
$c^A \, \bar{c}^B$	1	9	124	134
$\psi \, \bar{\psi}$	1	6	79	86
$A^A_\mu \bar{\psi} \, \psi$	2	33	697	732
Total	7	67	1182	1256

Table 2. Number of Feynman diagrams for each Green's function for the Curci-Ferrari gauge renormalization.

To deduce the renormalization constants themselves for each Green's functions, we apply the procedure discussed in [5]. Here one computes the Green's functions in terms of the bare parameters, g_{o}, α_{o}, $\bar{\alpha}_{\mathrm{o}}$ and ζ_{o}. The renormalized values are introduced by the definitions (3.1) and iteratively by loop order the renormalization constants are fixed by demanding that the overall infinity remaining is absorbed by the renormalization constant associated with that particular Green's function. As the Mincer algorithm is based on dimensional regularization in $d = 4 - 2\epsilon$ dimensions, we have absorbed all the poles in ϵ using the modified minimal subtraction scheme.

From a practical computing point of view we organised the one and two loop renormalization in a different way from the three loop computation. For the former we retained an arbitrary α, $\bar{\alpha}$ and ζ in the extraction of the renormalization constants. However, for the three loop case, due to the increase in the number of actual algebraic terms in a Feynman diagram to be evaluated, due to the presence of the parameters in the propagators and vertices, we chose to fix ζ to be 0 or 1 and $\bar{\alpha} = 0$ when the Feynman rules were substituted. This speeded up the computation significantly and avoided very large intermediate Form files which are generated. Running our code in the Landau gauge first allowed us to check the programme was performing correctly before generating the explicit value of the Feynman graph in the MAG. Moreover, at three loops one does not need to be concerned about the renormalization of the bare ζ in this approach since any corrections to this would only appear at three loops.

Having summarized the details of the computation we now record the explicit results. Rather than present the renormalization constants themselves, we have encoded them in the renormalization group functions defined by (3.6). Hence, with $\bar{\alpha} = \zeta = 0$, we find,

$$
\begin{aligned}
\gamma_A(a) \;=\; & \frac{1}{6N^o_A} \left[N^o_A \left((3\alpha - 13)C_A + 8T_F N_f \right) + N^d_A \left((-3\alpha + 9)C_A \right) \right] a \\
& + \frac{1}{48N^{o\,2}_A} \left[N^{o\,2}_A \left((6\alpha^2 + 66\alpha - 354)C_A^2 + 240C_A T_F N_f + 192 C_F T_F N_f \right) \right. \\
& \qquad + \; N^o_A N^d_A \left((3\alpha^2 + 210\alpha + 331)C_A^2 - 80 C_A T_F N_f \right) \\
& \qquad \left. + \; N^{d\,2}_A \left((15\alpha^2 - 6\alpha - 33)C_A^2 \right) \right] a^2 \\
& + \frac{1}{3456 N^{o\,3}_A} \left[N^{o\,3}_A ((162\alpha^3 + 2727\alpha^2 + 2592\alpha\zeta_3 + 18036\alpha + 1944\zeta_3 - 119580)C_A^3 \right. \\
& \qquad \quad + \; (-\,6912\alpha - 62208\zeta_3 + 174912)C_A^2 T_F N_f \\
& \qquad \quad + \; (82944\zeta_3 + 960)C_A C_F T_F N_f - 29184 C_A T_F^2 N_f^2 \\
& \qquad \quad - \; 6912 C_F^2 T_F N_f - 16896) C_F T_F^2 N_f^2)
\end{aligned}
$$

$$
\begin{aligned}
&+ \; N_A^{o2} N_A^d ((2133\alpha^3 + 162\alpha^2\zeta_3 + 25785\alpha^2 + 14904\alpha\zeta_3 + 61479\alpha \\
&\qquad\qquad -\; 3564\zeta_3 + 105550) C_A^3 \\
&\qquad\qquad +\; (-\,13392\alpha - 62208\zeta_3 - 31264) C_A^2 T_F N_f \\
&\qquad\qquad +\; (82944\zeta_3 - 77760) C_A C_F T_F N_f - 8960 C_A T_F^2 N_f^2) \\
&+ \; N_A^o N_A^{d2} ((-324\alpha^3\zeta_3 + 1728\alpha^3 - 6480\alpha^2\zeta_3 + 14256\alpha^2 \\
&\qquad\qquad - 11988\alpha\zeta_3 - 26298\alpha - 129924\zeta_3 - 113751) C_A^3 \\
&\qquad\qquad + (13392\alpha + 41472\zeta_3 + 9936) C_A^2 T_F N_f) \\
&+ N_A^{d3} ((-\,4536\alpha^3\zeta_3 - 270\alpha^3 - 18792\alpha^2\zeta_3 - 3294\alpha^2 - 82296\alpha\zeta_3 \\
&\qquad\qquad + 42714\alpha - 176904\zeta_3 + 101952) C_A^3) \Big] a^3 \; + \; O(a^4) \quad (3.7)
\end{aligned}
$$

where ζ_n is the Riemann zeta function and

$$
\begin{aligned}
\gamma_\alpha(a) \;=\; & \frac{1}{12\alpha N_A^o} \Big[N_A^o \Big((-\,3\alpha^2 + 26\alpha) C_A - 16\alpha T_F N_f \Big) + N_A^d \Big((-\,6\alpha^2 - 36\alpha - 36) C_A \Big) \Big] a \\
& + \frac{1}{48\alpha N_A^{o2}} \Big[N_A^{o2} \Big((-\,3\alpha^3 - 51\alpha^2 + 354\alpha) C_A^2 - 240\alpha C_A T_F N_f - 192\alpha C_F T_F N_f \Big) \\
&\qquad + \; N_A^o N_A^d \Big((-\,27\alpha^3 - 339\alpha^2 - 647\alpha - 928) C_A^2 + (160\alpha + 512) C_A T_F N_f \Big) \\
&\qquad + \; N_A^{d2} \Big((-\,30\alpha^3 - 366\alpha^2 + 294\alpha + 2016) C_A^2 \Big) \Big] a^2 \\
& + \frac{1}{6912\alpha N_A^{o3}} \Big[N_A^{o3} ((-\,162\alpha^4 - 3348\alpha^3 - 5184\alpha^2\zeta_3 - 25218\alpha^2 \\
&\qquad\qquad -\; 3888\alpha\zeta_3 + 239160\alpha) C_A^3 \\
&\qquad\qquad +\; (7344\alpha^2 + 124416\alpha\zeta_3 - 349824\alpha) C_A^2 T_F N_f \\
&\qquad\qquad +\; (-\,165888\alpha\zeta_3 - 1920\alpha) C_A C_F T_F N_f + 58368\alpha C_A T_F^2 N_f^2 \\
&\qquad\qquad +\; 13824\alpha C_F^2 T_F N_f + 33792\alpha C_F T_F^2 N_f^2) \\
&\qquad + \; N_A^{o2} N_A^d ((-\,2754\alpha^4 - 48492\alpha^3 - 14256\alpha^2\zeta_3 - 155493\alpha^2 \\
&\qquad\qquad +\; 27864\alpha\zeta_3 - 256744\alpha + 209952\zeta_3 - 548904) C_A^3 \\
&\qquad\qquad +\; (29376\alpha^2 + 207360\alpha\zeta_3 + 36064\alpha \\
&\qquad\qquad\quad +\; 331776\zeta_3 + 136128) C_A^2 T_F N_f \\
&\qquad\qquad +\; (-\,331776\alpha\zeta_3 + 311040\alpha \\
&\qquad\qquad\quad -\; 663552\zeta_3 + 705024) C_A C_F T_F N_f \\
&\qquad\qquad +\; (35840\alpha + 61440) C_A T_F^2 N_f^2) \\
&\qquad + \; N_A^o N_A^{d2} ((-\,7884\alpha^4 - 133920\alpha^3 + 76464\alpha^2\zeta_3 - 151524\alpha^2 \\
&\qquad\qquad +\; 517752\alpha\zeta_3 + 503388\alpha + 1666656\zeta_3 + 1014012) C_A^3 \\
&\qquad\qquad +\; (29376\alpha^2 - 248832\alpha\zeta_3 + 5184\alpha \\
&\qquad\qquad -\; 995328\zeta_3 - 812160) C_A^2 T_F N_f) \\
&\qquad + \; N_A^{d3} ((-\,6480\alpha^4 - 105840\alpha^3 - 220320\alpha^2\zeta_3 + 110700\alpha^2 \\
&\qquad\qquad -\; 784080\alpha\zeta_3 + 373032\alpha \\
&\qquad\qquad -\; 1021248\zeta_3 - 3148632) C_A^3) \Big] a^3 \; + \; O(a^4) \; . \qquad (3.8)
\end{aligned}
$$

For completeness we record the sum of the previous two anomalous dimensions partly to indicate the singular nature of this renormalization group function, but also because it corresponds to the renormalization of the gauge parameter itself from the convention we have used to define it.

We have

$$
\begin{aligned}
\gamma_A(a) \;+\; \gamma_\alpha(a) \;&=\; \frac{C_A}{4\alpha N^o_A}\left[\alpha^2 N^o_A - \left(4\alpha^2+6\alpha+12\right) N^d_A\right] a \\
&+\; \frac{C_A}{48\alpha N^{o2}_A}\left[N^{o2}_A\left((3\alpha^3+15\alpha^2)C_A\right)\right. \\
&\qquad +\; N^o_A N^d_A\left((-\,24\alpha^3-129\alpha^2-316\alpha-928)C_A\right. \\
&\qquad\qquad +\; \left.(80\alpha+512)T_F N_f\right) \\
&\qquad +\; \left. N^{d2}_A\left((-\,15\alpha^3-372\alpha^2+261\alpha+2016)C_A\right)\right] a^2 \\
&+\; \frac{1}{6912\alpha N^{o3}_A}\left[N^{o3}_A((162\alpha^4+2106\alpha^3+10854\alpha^2)C^3_A - 6480\alpha^2 C^2_A T_F N_f)\right. \\
&\qquad +\; N^{o2}_A N^d_A((1512\alpha^4+324\alpha^3\zeta_3+3078\alpha^3+15552\alpha^2\zeta_3 \\
&\qquad\qquad -\; 32535\alpha^2+20736\alpha\zeta_3-45644\alpha \\
&\qquad\qquad +\; 209952\zeta_3-548904)C^3_A \\
&\qquad\qquad +\; (2592\alpha^2+82944\alpha\zeta_3-26464\alpha \\
&\qquad\qquad\quad +\; 331776\zeta_3+136128)C^2_A T_F N_f \\
&\qquad\qquad +\; (-\,165888\alpha\zeta_3+155520\alpha \\
&\qquad\qquad\quad -\; 663552\zeta_3+705024)C_A C_F T_F N_f \\
&\qquad\qquad +\; (17920\alpha+61440)C_A T^2_F N^2_f) \\
&\qquad +\; N^o_A N^{d2}_A((-\,648\alpha^4\zeta_3-4428\alpha^4-12960\alpha^3\zeta_3 \\
&\qquad\qquad -\; 105408\alpha^3+52488\alpha^2\zeta_3-204120\alpha^2 \\
&\qquad\qquad +\; 15552\alpha^2\zeta_3+257904\alpha\zeta_3+275886\alpha \\
&\qquad\qquad +\; 1666656\zeta_3+1014012)C^3_A \\
&\qquad\qquad +\; (56160\alpha^2-165888\alpha\zeta_3+25056\alpha \\
&\qquad\qquad\quad -\; 995328\zeta_3-812160)C^2_A T_F N_f) \\
&\qquad +\; N^{d3}_A((-\,9072\alpha^4\zeta_3-7020\alpha^4-37584\alpha^3\zeta_3-112428\alpha^3 \\
&\qquad\qquad -\; 384912\alpha^2\zeta_3+196128\alpha^2-1137888\alpha\zeta_3 \\
&\qquad\qquad +\; \left.576936\alpha-1021248\zeta_3-3148632)C^3_A)\right] a^3 \\
&+\; O(a^4)\;.
\end{aligned} \tag{3.9}
$$

Next,

$$
\begin{aligned}
\gamma_{A^i}(a) \;&=\; \frac{1}{3}\left[4T_F N_f - 11C_A\right] a \\
&+\; \frac{1}{3}\left[-\,34C^2_A+20C_A T_F N_f+12C_F T_F N_f\right] a^2 \\
&+\; \frac{1}{54}\left[-\,2857C^3_A+2830C^2_A T_F N_f+1230C_A C_F T_F N_f\right. \\
&\qquad -\; \left.316C_A T^2_F N^2_f-108C^2_F T_F N_f-264C_F T^2_F N^2_f\right] a^3 \;+\; O(a^4)
\end{aligned} \tag{3.10}
$$

and we have checked explicitly that when $\bar{\alpha}=0$

$$
\gamma_{\alpha^i}(a) \;=\; -\;\gamma_{A^i}(a) \;+\; O(a^4)\;. \tag{3.11}
$$

For the ghosts and quarks we have

$$
\gamma_c(a) \;=\; \frac{1}{4N^o_A}\left[N^o_A\left((\alpha-3)C_A\right)+N^d_A\left((-2\alpha-6)C_A\right)\right] a
$$

$$
\begin{aligned}
& + \frac{1}{96N_A^{o2}} \Big[N_A^{o2} \Big((6\alpha^2 - 6\alpha - 190)C_A^2 + 80C_AT_FN_f \Big) \\
& \qquad + N_A^o N_A^d \Big((-42\alpha^2 - 126\alpha - 347)C_A^2 + 160C_AT_FN_f \Big) \\
& \qquad + N_A^{d2} \Big((12\alpha^2 - 588\alpha + 510)C_A^2 \Big) \Big] a^2 \\
& + \frac{1}{6912N_A^{o3}} \Big[N_A^{o3}((162\alpha^3 + 1485\alpha^2 - 2592\alpha\zeta_3 + 3672\alpha - 1944\zeta_3 - 63268)C_A^3 \\
& \qquad + (-6048\alpha + 62208\zeta_3 + 6208)C_A^2T_FN_f \\
& \qquad + (-82944\zeta_3 + 77760)C_AC_FT_FN_f + 8960C_AT_F^2N_f^2) \\
& \qquad + N_A^{o2}N_A^d((1242\alpha^3 + 10287\alpha^2 + 8748\alpha\zeta_3 + 2565\alpha \\
& \qquad \quad + 57996\zeta_3 - 19184)C_A^3 \\
& \qquad \quad + (-5616\alpha + 103680\zeta_3 - 47632)C_A^2T_FN_f \\
& \qquad \quad + (-165888\zeta_3 + 155520)C_AC_FT_FN_f + 17920C_AT_F^2N_f^2) \\
& \qquad + N_A^o N_A^{d2}((-1296\alpha^3\zeta_3 - 1836\alpha^3 - 16200\alpha^2\zeta_3 - 68148\alpha^2 \\
& \qquad \quad + 140292\alpha\zeta_3 - 161730\alpha + 617868\zeta_3 - 258174)C_A^3 \\
& \qquad \quad + (35424\alpha - 124416\zeta_3 - 18144)C_A^2T_FN_f) \\
& \qquad + N_A^{d3}((-18144\alpha^3\zeta_3 + 864\alpha^3 + 11664\alpha^2\zeta_3 - 89532\alpha^2 - 191160\alpha\zeta_3 \\
& \qquad \quad + 128304\alpha - 21384\zeta_3 - 135972)C_A^3) \Big] a^3 + O(a^4)
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
\gamma_{c^i}(a) &= \frac{1}{4N_A^o} \Big[N_A^o ((-\alpha - 3)C_A) + N_A^d ((-2\alpha - 6)C_A) \Big] a \\
& + \frac{1}{96N_A^{o2}} \Big[N_A^{o2} \Big((-6\alpha^2 - 66\alpha - 190)C_A^2 + 80C_AT_FN_f \Big) \\
& \qquad + N_A^o N_A^d \Big((-54\alpha^2 - 354\alpha - 323)C_A^2 + 160C_AT_FN_f \Big) \\
& \qquad + N_A^{d2} \Big((-60\alpha^2 - 372\alpha + 510)C_A^2 \Big) \Big] a^2 \\
& + \frac{1}{6912N_A^{o3}} \Big[N_A^{o3}((-162\alpha^3 - 2727\alpha^2 - 2592\zeta_3\alpha - 18036\alpha - 1944\zeta_3 - 63268)C_A^3 \\
& \qquad + (6912\alpha + 62208\zeta_3 + 6208)C_A^2T_FN_f \\
& \qquad + (-82944\zeta_3 + 77760)C_AC_FT_FN_f + 8960C_AT_F^2N_f^2) \\
& \qquad + N_A^{o2}N_A^d((-2754\alpha^3 + 648\zeta_3\alpha^2 - 28917\alpha^2 - 4212\zeta_3\alpha \\
& \qquad \quad - 69309\alpha + 37260\zeta_3 - 64544)C_A^3 \\
& \qquad \quad + (25488\alpha + 103680\zeta_3 - 13072)C_A^2T_FN_f \\
& \qquad \quad + (-165888\zeta_3 + 155520)C_AC_FT_FN_f + 17920C_AT_F^2N_f^2) \\
& \qquad + N_A^o N_A^{d2}((-7884\alpha^3 + 22680\zeta_3\alpha^2 - 84564\alpha^2 + 97524\zeta_3\alpha \\
& \qquad \quad - 47142\alpha + 433836\zeta_3 - 56430)C_A^3 \\
& \qquad \quad + (25056\alpha - 124416\zeta_3 - 18144)C_A^2T_FN_f) \\
& \qquad + N_A^{d3}((-6480\alpha^3 + 34992\zeta_3\alpha^2 - 70092\alpha^2 + 8424\zeta_3\alpha \\
& \qquad \quad + 114912\alpha + 77112\zeta_3 - 161028)C_A^3) \Big] a^3 + O(a^4)
\end{aligned} \tag{3.13}
$$

and

$$
\gamma_\psi(a) = \frac{\alpha N_A^o T_F}{N_F} a
$$

$$
\begin{aligned}
&+ \frac{1}{4N_F}\Big[(-\,\alpha^2+22\alpha+23)C_AC_FN_F+(\alpha^2-14\alpha+2)N^o_AC_AT_F \\
&\qquad -\;6C_F^2N_F-8C_FN_fT_FN_F\Big]\,a^2 \\
&+ \frac{1}{576N_FT_FN^o_A}\Big[(684\alpha^3+(1296\zeta_3+3528)\alpha^2+(22464\zeta_3-14094)\alpha \\
&\qquad +\;30240\zeta_3-33264)C_A^2C_F^2N_F^2 \\
&\qquad +\;(-\,810\alpha^3+(-2376\zeta_3-1908)\alpha^2+(-40608\zeta_3+42039)\alpha \\
&\qquad\quad -\;63072\zeta_3+109016)C_A^2C_FN_FN^o_AT_F \\
&\qquad +\;(180\alpha^3+(1080\zeta_3-1080)\alpha^2+(18576\zeta_3-23211)\alpha \\
&\qquad\quad +\;27864\zeta_3-39132)C_A^2N^{o\,2}_AT_F^2 \\
&\qquad +\;(-\,5472\alpha-24128)C_AC_FN_FN^o_AN_fT_F^2 \\
&\qquad +\;(3024\alpha+5760)C_AN^{o\,2}_AN_fT_F^3 \\
&\qquad +\;(6912\zeta_3-20592)C_AC_F^2N_FN^o_AT_F \\
&\qquad +\;1728C_F^2N_FN^o_AN_fT_F^2+1280C_FN_FN^o_AN_f^2T_F^3 \\
&\qquad +\;864C_F^3N_FN^o_AT_F\Big]\,a^3\;+\;O(a^4)\;. \qquad (3.14)
\end{aligned}
$$

Finally, for completeness we record that the β-function emerges as

$$
\begin{aligned}
\beta(a) \;=\;& -\;\left[\frac{11}{3}C_A-\frac{4}{3}T_FN_f\right]a^2\;-\;\left[\frac{34}{3}C_A^2-4C_FT_FN_f-\frac{20}{3}C_AT_FN_f\right]a^3 \\
&+\;\Big[2830C_A^2T_FN_f-2857C_A^3+1230C_AC_FT_FN_f-316C_AT_F^2N_f^2 \\
&\qquad -\;108C_F^2T_FN_f-264C_FT_F^2N_f^2\Big]\frac{a^4}{54}\;+\;O(a^5)\;. \qquad (3.15)
\end{aligned}
$$

To have confidence in the correctness of these results it is important to indicate the checks we have carried out. First, from a renormalization point of view, using the method of [5] the double and triple poles in ϵ at three loops and the double pole at two loops in the renormalization constants are not independent of the previous order one loop poles. Therefore, we have checked that these emerge correctly for both the Landau gauge and the MAG. This is a non-trivial observation given the particular structure of the renormalization group functions which depend not only on α but also on the colour group Casimirs and for the MAG, the dimensions of the centre and off-diagonal sector of the Lie group. Second, the β-function correctly emerges from the renormalization of the centre gluon. Again this is non-trivial since in the renormalization of the 2-point function its divergence has to emerge to be independent of not only α but also of N^d_A and N^o_A as well as being equivalent to the actual β-function itself. By the same token we can of course trivially record that the four loop anomalous dimension for A^i_μ is, [6],

$$
\begin{aligned}
\gamma_{A^i}(a) \;=\;& \frac{1}{3}\left[4T_FN_f-11C_A\right]a \\
&+\frac{1}{3}\Big[-\,34C_A^2+20C_AT_FN_f+12C_FT_FN_f\Big]\,a^2 \\
&+\frac{1}{54}\Big[-\,2857C_A^3+2830C_A^2T_FN_f+1230C_AC_FT_FN_f \\
&\qquad -\;316C_AT_F^2N_f^2-108C_F^2T_FN_f-264C_FT_F^2N_f^2\Big]\,a^3 \\
&+\;\Bigg[\left(\frac{44}{9}\zeta_3-\frac{150653}{486}\right)C_A^4+\left(\frac{39143}{81}-\frac{136}{3}\zeta_3\right)C_A^3T_FN_f \\
&\quad +\left(\frac{656}{9}\zeta_3-\frac{7073}{243}\right)C_A^2C_FT_FN_f+\left(\frac{4204}{27}-\frac{352}{9}\zeta_3\right)C_AC_F^2T_FN_f-46C_F^3T_FN_f
\end{aligned}
$$

$$
\begin{aligned}
& - \left(\frac{7930}{81} + \frac{224}{9}\zeta_3 \right) C_A^2 T_F^2 N_f^2 + \left(\frac{704}{9}\zeta_3 - \frac{1352}{27} \right) C_F^2 T_F^2 N_f^2 \\
& - \left(\frac{17152}{243} + \frac{448}{9}\zeta_3 \right) C_A C_F T_F^2 N_f^2 - \frac{424}{243} C_A T_F^3 N_f^3 - \frac{1232}{243} C_F T_F^3 N_f^3 \\
& + \left(\frac{80}{9} - \frac{704}{3}\zeta_3 \right) \frac{d_A^{ABCD} d_A^{ABCD}}{N_F} + \left(\frac{1664}{3}\zeta_3 - \frac{512}{9} \right) \frac{d_F^{ABCD} d_A^{ABCD}}{N_F} N_f \\
& + \left(\frac{704}{9} - \frac{512}{3}\zeta_3 \right) \frac{d_F^{ABCD} d_F^{ABCD}}{N_F} N_f^2 \Bigg] a^5 \; + \; O(a^6)
\end{aligned}
\tag{3.16}
$$

where

$$
d_F^{ABCD} \;=\; \mathrm{Tr} \left(T^A T^{(B} T^C T^{D)} \right) \tag{3.17}
$$

and d_A^{ABCD} is d_F^{ABCD} evaluated in the adjoint representation.

The next checks concern the anomalous dimensions themselves in certain limits. We have already indicated that the programmes we have used correctly reproduce all the Landau gauge results prior to switching to the MAG. However, the anomalous dimensions are also related to those of the Curci-Ferrari gauge. For instance, for the off-diagonal gluon, off-diagonal ghost and quark, taking the formal limit $N_A^d/N_A^o \to 0$, then the following anomalous dimensions arise for *arbitrary* α,

$$
\begin{aligned}
\lim_{N_A^d/N_A^o \to 0} \gamma_A(a) \;=\; & \frac{1}{6} \left[(3\alpha - 13) C_A + 8 T_F N_f \right] a \\
& + \frac{1}{48} \left[(6\alpha^2 + 66\alpha - 354) C_A^2 + 240 C_A T_F N_f + 192 C_F T_F N_f \right] a^2 \\
& + \frac{1}{3456} \Big[(162\alpha^3 + 2727\alpha^2 + 2592\alpha\zeta_3 + 18036\alpha + 1944\zeta_3 - 119580) C_A^3 \\
& \quad + \; (-\, 6912\alpha - 62208\zeta_3 + 174912) C_A^2 T_F N_f \\
& \quad + \; (82944\zeta_3 + 960) C_A C_F T_F N_f - 29184 C_A T_F^2 N_f^2 \\
& \quad - \; 6912 C_F^2 T_F N_f - 16896) C_F T_F^2 N_f^2 \Big] a^3 \; + \; O(a^4)
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\lim_{N_A^d/N_A^o \to 0} \gamma_\alpha(a) \;=\; & \frac{1}{12} \left[(-3\alpha + 26) C_A - 16 T_F N_f \right] a \\
& + \frac{1}{48} \left[(-\, 3\alpha^2 - 51\alpha + 354) C_A^2 - 240 C_A T_F N_f - 192 C_F T_F N_f \right] a^2 \\
& + \frac{1}{6912} \Big[(-\, 162\alpha^3 - 3348\alpha^2 - 5184\alpha\zeta_3 - 25218\alpha \\
& \quad - \; 3888\zeta_3 + 239160) C_A^3 \\
& \quad + \; (7344\alpha + 124416\zeta_3 - 349824) C_A^2 T_F N_f \\
& \quad + \; (-\, 165888\zeta_3 - 1920) C_A C_F T_F N_f + 58368 C_A T_F^2 N_f^2 \\
& \quad + \; 13824 C_F^2 T_F N_f + 33792 C_F T_F^2 N_f^2 \Big] a^3 \; + \; O(a^4)
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\lim_{N_A^d/N_A^o \to 0} \left[\gamma_A(a) \; + \; \gamma_\alpha(a) \right] \;=\; & \frac{\alpha C_A}{4} a + \frac{\alpha(\alpha + 5) C_A^2}{16} a^2 \\
& + \frac{3\alpha}{128} \left[(\alpha^2 + 13\alpha + 67) C_A - 40 T_F N_f \right] C_A^2 a^3 \; + \; O(a^4)
\end{aligned}
\tag{3.20}
$$

$$
\lim_{N_A^d/N_A^o \to 0} \gamma_c(a) \;=\; \frac{1}{4} \left[(\alpha - 3) C_A \right] a
$$

$$
\begin{aligned}
& + \frac{1}{96}\left[(6\alpha^2 - 6\alpha - 190)C_A^2 + 80C_AT_FN_f\right]a^2 \\
& + \frac{1}{6912}\left[(162\alpha^3 + 1485\alpha^2 - 2592\alpha\zeta_3 + 3672\alpha - 1944\zeta_3 - 63268)C_A^3\right. \\
& \qquad + (-6048\alpha + 62208\zeta_3 + 6208)C_A^2T_FN_f \\
& \qquad \left. + (-82944\zeta_3 + 77760)C_AC_FT_FN_f + 8960C_AT_F^2N_f^2)\right]a^3 \\
& + O(a^4)
\end{aligned} \tag{3.21}
$$

and

$$
\begin{aligned}
\lim_{N_A^d/N_A^o \to 0} \gamma_\psi(a) &= \frac{\alpha C_F}{4}a + \frac{1}{4}\left[(8\alpha + 25)C_AC_F - 6C_F^2 - 8C_FT_FN_f\right]a^2 \\
& + \frac{1}{288}\left[(27\alpha^3 + 270\alpha^2 + 216\alpha\zeta_3 + 2367\alpha - 2484\zeta_3 + 18310)C_A^2C_F\right. \\
& \qquad + (-1224\alpha - 9184)C_AC_FT_FN_f + 432C_F^3 + 864C_FT_FN_f \\
& \qquad \left. + (3456\zeta_3 - 10296)C_AC_F^2 + 640C_FT_F^2N_f^2)\right]a^3 + O(a^4)
\end{aligned} \tag{3.22}
$$

where to take the limit for the quark anomalous dimension* we have used the result that

$$
\lim_{N_A^d/N_A^o \to 0} \frac{T_FN_A^o}{N_F} = C_F \; . \tag{3.23}
$$

Comparing these limits with [12, 51, 52], we observe that they are equivalent to the three loop $\overline{\mathrm{MS}}$ anomalous dimensions in the Curci-Ferrari gauge for arbitrary α. That this result appears is not unexpected since Kondo indicated in [19] that the off-diagonal sector is in fact the Curci-Ferrari gauge. Indeed this observation, and its relation to the generation of a non-zero vacuum expectation value for the operator $\mathcal{O}$, was one of the reasons for the recent renewed interest in both the Curci-Ferrari gauge and MAG. That the Curci-Ferrari anomalous dimensions correctly emerge is an important check on the full MAG computation. A final more trivial check rests in taking the formal abelian limit in the Landau gauge, $C_A \to 0$, $C_F \to 1$, $T_F \to 1$ and $\alpha \to 0$. One observes that both ghost anomalous dimensions vanish, the centre and off-diagonal gluon anomalous dimensions reduce to the quantum electrodynamics β-function and the quark anomalous dimension tends to the electron anomalous dimension.

Having justified the results for the full renormalization of the QCD Lagrangian in the MAG, we can now deduce the anomalous dimension of $\mathcal{O}$. At two loops we actually computed the anomalous dimension directly by the same method as [12]. The operator was inserted in an off-diagonal ghost 2-point function and the corresponding renormalization constant $Z_{\mathcal{O}}$ was extracted. Computing the associated anomalous dimension directly from the renormalization constant, the resulting two loop value correctly satisfied (3.3). At three loops we took the point of view that the three loop $\gamma_{c^i}(a)$ was correctly determined and therefore used (3.3) to deduce

$$
\begin{aligned}
\gamma_{\mathcal{O}}(a) &= \frac{1}{12N_A^o}\left[N_A^o((-3\alpha + 35)C_A - 16T_fN_f) + N_A^d((-6\alpha - 18)C_A)\right]a \\
& + \frac{1}{96N_A^{o2}}\left[N_A^{o2}\left((-6\alpha^2 - 66\alpha + 898)C_A^2 - 560C_AT_fN_f - 384C_FT_fN_f\right)\right. \\
& \qquad + N_A^oN_A^d\left((-54\alpha^2 - 354\alpha - 323)C_A^2 + 160C_AT_fN_f\right) \\
& \qquad \left. + N_A^{d2}\left((-60\alpha^2 - 372\alpha + 510)C_A^2\right)\right]a^2
\end{aligned}
$$

*Whilst the renormalization constant for Z_ψ was explicity given in [12] for the Curci-Ferrari gauge, the actual anomalous dimension was inadvertently omitted.

$$
\begin{aligned}
&+ \frac{1}{6912 N_A^{o3}} \Big[N_A^{o3} ((-\ 162\alpha^3 - 2727\alpha^2 - 2592\zeta_3\alpha - 18036\alpha - 1944\zeta_3 + 302428) C_A^3 \\
&\qquad + \ (6912\alpha + 62208\zeta_3 - 356032) C_A^2 T_F N_f \\
&\qquad + \ (-\ 82944\zeta_3 - 79680) C_A C_F T_F N_f + 49408 C_A T_F^2 N_f^2 \\
&\qquad + \ 13824 C_F^2 T_F N_f + 33792 C_F T_F^2 N_f^2) \\
&\quad + \ N_A^{o2} N_A^d ((-\ 2754\alpha^3 + 648\alpha^2\zeta_3 - 28917\alpha^2 - 4212\alpha\zeta_3 \\
&\qquad\qquad - \ 69309\alpha + 37260\zeta_3 - 64544) C_A^3 \\
&\qquad\qquad + \ (25488\alpha + 103680\zeta_3 - 13072) C_A^2 T_F N_f \\
&\qquad\qquad + \ (-\ 165888\zeta_3 + 155520) C_A C_F T_F N_f + 17920 C_A T_F^2 N_f^2) \\
&\quad + \ N_A^o N_A^{d2} ((-\ 7884\alpha^3 + 22680\alpha^2\zeta_3 - 84564\alpha^2 + 97524\alpha\zeta_3 - 47142\alpha \\
&\qquad\qquad + \ 433836\zeta_3 - 56430) C_A^3 \\
&\qquad\qquad + \ (25056\alpha - 124416\zeta_3 - 18144) C_A^2 T_F N_f) \\
&\quad + \ N_A^{d3} ((-\ 6480\alpha^3 + 34992\alpha^2\zeta_3 - 70092\alpha^2 + 8424\alpha\zeta_3 + 114912\alpha \\
&\qquad\qquad + \ 77112\zeta_3 - 161028) C_A^3) \Big] a^3 \ + \ O(a^4) \ . \qquad (3.24)
\end{aligned}
$$

Again there are several checks on this result aside from the internal renormalization group consistency check. First, we verified that the Landau gauge anomalous dimension emerged correctly. Second, in the formal limit $N_A^d/N_A^o \to 0$ $\gamma_{\mathcal{O}}(a)$ tends to the Curci-Ferrari gauge result for all α, [12, 51, 52].

Finally, having derived the anomalous dimensions of all the fields and $\mathcal{O}$ for an arbitrary colour group, we record the explicit results for the two main Lie groups of interest. For $SU(2)$ with $C_A = 2$, $C_F = 3/4$, $T_F = 1/2$, $N_F = 2$, $N_A^o = 2$ and $N_A^d = 1$, we have

$$
\begin{aligned}
\gamma_A(a) \ &= \ [3\alpha - 17 + 4N_f] \frac{a}{6} \ + \ \left[45\alpha^2 + 678\alpha - 787 + 272N_f\right] \frac{a^2}{48} \\
&\quad + \Big[-\ 2592\alpha^3\zeta_3 + 6507\alpha^3 - 15552\alpha^2\zeta_3 + 75087\alpha^2 - 10260\alpha N_f \\
&\quad - \ 12960\alpha\zeta_3 + 190161\alpha - 10000N_f^2 - 36288N_f\zeta_3 + 146572N_f \\
&\quad - \ 217728\zeta_3 - 329995\Big] \frac{a^3}{1728} \ + \ O(a^4) \\
\gamma_\alpha(a) \ &= \ \left[-\ 3\alpha^2 + 4\alpha - 9 - 2\alpha N_f\right] \frac{a}{3\alpha} \\
&\quad + \left[-\ 12\alpha^3 - 156\alpha^2 + 52\alpha + 20 - (29\alpha - 32)N_f\right] \frac{a^2}{6\alpha} \\
&\quad + \Big[-\ 2160\alpha^4 - 37152\alpha^3 + 3672\alpha^2 N_f - \ 10368\alpha^2\zeta_3 - 63504\alpha^2 \\
&\quad + \ 2780\alpha N_f^2 + 5184\alpha N_f\zeta_3 - 33869\alpha N_f + 20736\alpha\zeta_3 + 141632\alpha \\
&\quad + \ 960N_f^2 - 25920N_f\zeta_3 - 348N_f + 196992\zeta_3 - 207264\Big] \frac{a^3}{432\alpha} \ + \ O(a^4) \\
\gamma_A(a) \ + \ \gamma_\alpha(a) \ &= \ - \ \left[\alpha^2 + 3\alpha + 6\right] \frac{a}{2\alpha} \\
&\quad + \left[-\ 51\alpha^3 - 570\alpha^2 - 371\alpha + 160 + (40\alpha + 256)N_f\right] \frac{a^2}{48\alpha} \\
&\quad + \Big[-\ 2592\alpha^4\zeta_3 - 2133\alpha^4 - 15552\alpha^3\zeta_3 - 73521\alpha^3 + 4428\alpha^2 N_f \\
&\quad - \ 54432\alpha^2\zeta_3 - 63855\alpha^2 + 1120\alpha N_f^2 - 15552\alpha N_f\zeta_3 + 11096\alpha N_f \\
&\quad - \ 134784\alpha\zeta_3 + 236533\alpha + 3840N_f^2 - 103680N_f\zeta_3 - 1392N_f
\end{aligned}
$$

$$
\begin{aligned}
&+\ 787968\zeta_3 - 829056] \frac{a^3}{1728\alpha} \ +\ O(a^4) \\
\gamma_{A^i}(a) &= 2\left[N_f - 11\right] \frac{a}{3} \ +\ \left[49N_f - 272\right] \frac{a^2}{6} \\
&+ \left[-1660N_f^2 + 52417N_f - 182848\right] \frac{a^3}{432} \ +\ O(a^4) \\
\gamma_c(a) &= -\ 3\alpha\, a \ +\ \left[-\ 3\alpha^2 - 54\alpha - 59 + 10N_f\right] \frac{a^2}{6} \\
&+ \left[-\ 648\alpha^3\zeta_3 + 108\alpha^3 - 648\alpha^2\zeta_3 - 5400\alpha^2 + 3240\alpha\zeta_3 - 4860\alpha + 280N_f^2\right. \\
&\left. +\ 1296N_f\zeta_3 + 2261N_f + 44712\zeta_3 - 38600\right] \frac{a^3}{216} \ +\ O(a^4) \\
\gamma_{c^i}(a) &= -\ [\alpha + 3]\, a \ +\ \left[-\ 6\alpha^2 - 42\alpha - 28 + 5N_f\right] \frac{a^2}{3} \\
&+ \left[-\ 1080\alpha^3 + 2592\alpha^2\zeta_3 - 11772\alpha^2 + 1620\alpha N_f + 5184\alpha\zeta_3 - 12528\alpha\right. \\
&\left. +\ 280N_f^2 + 1296N_f\zeta_3 + 3341N_f + 33696\zeta_3 - 32444\right] \frac{a^3}{216} \ +\ O(a^4) \\
\gamma_\psi(a) &= \frac{1}{2}\alpha\, a \ +\ \left[-\ 4\alpha^2 + 152\alpha + 265 - 24N_f\right] \frac{a^2}{32} \ +\ O(a^3) \\
&+ \left[672\alpha^3 + 576\alpha^2\zeta_3 + 5328\alpha^2 - 1728\alpha N_f + 10944\alpha\zeta_3 + 10848\alpha\right. \\
&\left. +\ 160N_f^2 - 9820N_f + 6912\zeta_3 + 50863\right] \frac{a^3}{384} \ +\ O(a^4) \\
\gamma_{\mathcal{O}}(a) &= [-\ 3\alpha + 13 - 2N_f] \frac{a}{3} \ +\ \left[-\ 4\alpha^2 - 28\alpha + 72 - 13N_f\right] \frac{a^2}{2} \\
&+ \left[-\ 720\alpha^3 + 1728\alpha^2\zeta_3 - 7848\alpha^2 + 1080\alpha N_f + 3456\alpha\zeta_3 - 8352\alpha\right. \\
&\left. +\ 740N_f^2 + 864N_f\zeta_3 - 15245N_f + 22464\zeta_3 + 39320\right] \frac{a^3}{144} \ +\ O(a^4) \ .
\end{aligned}
\tag{3.25}
$$

The one loop expressions agree with the limited known results. Repeating the same exercise for $SU(3)$ with $C_A = 3$, $C_F = 4/3$, $T_F = 1/2$, $N_F = 3$, $N_A^o = 6$ and $N_A^d = 2$, we have

$$
\begin{aligned}
\gamma_A(a) &= [3\alpha - 15 + 2N_f] \frac{a}{3} \ +\ \left[39\alpha^2 + 609\alpha - 1113 + 224N_f\right] \frac{a^2}{24} \\
&+ \left[-\ 1836\alpha^3\zeta_3 + 9495\alpha^3 - 12258\alpha^2\zeta_3 + 115056\alpha^2 - 14832\alpha N_f\right. \\
&+\ 28620\alpha\zeta_3 + 334701\alpha - 9920N_f^2 - 43776N_f\zeta_3 + 229704N_f \\
&\left. -\ 182088\zeta_3 - 839337\right] \frac{a^3}{1152} \ +\ O(a^4) \\
\gamma_\alpha(a) &= \left[-\ 15\alpha^2 + 42\alpha - 36 - 8\alpha N_f\right] \frac{a}{12\alpha} \\
&+ \left[-\ 138\alpha^3 - 1842\alpha^2 + 1539\alpha - 768 - (408\alpha - 256)N_f\right] \frac{a^2}{48\alpha} \\
&+ \left[-\ 59292\alpha^4 - 1034424\alpha^3 + 91800\alpha^2 N_f - 259200\alpha^2\zeta_3 - 2424195\alpha^2\right. \\
&+\ 64000\alpha N_f^2 + 193536\alpha N_f\zeta_3 - 1301712\alpha N_f + 914976\alpha\zeta_3 \\
&+\ 6029820\alpha + 15360N_f^2 - 442368N_f\zeta_3 + 268128N_f \\
&\left. +\ 5868288\zeta_3 - 5046732\right] \frac{a^3}{6912\alpha} \ +\ O(a^4)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_A(a) \; + \; \gamma_\alpha(a) \;\; &= \;\; - \; \left[\alpha^2 + 6\alpha + 12 \right] \frac{a}{4\alpha} \\
&+ \; \left[- \; 60\alpha^3 - 624\alpha^2 - 687\alpha - 768 + (40\alpha + 256) N_f \right] \frac{a^2}{48\alpha} \; + \; O(a^3) \\
&+ \; \left[- \; 11016\alpha^4\zeta_3 - 2322\alpha^4 - 73548\alpha^3\zeta_3 - 344088\alpha^3 + 2808\alpha^2 N_f \right. \\
&\quad - \; 87480\alpha^2\zeta_3 - 415989\alpha^2 + 4480\alpha N_f^2 - 69120\alpha N_f\zeta_3 + 76512\alpha N_f \\
&\quad - \; 177552\alpha\zeta_3 + 993798\alpha + 15360 N_f^2 - 442368 N_f\zeta_3 + 268128 N_f \\
&\quad \left. + \; 5868288\zeta_3 - 5046732 \right] \frac{a^3}{6912\alpha} \; + \; O(a^4) \\
\gamma_{A^i}(a) \;\; &= \;\; [2N_f - 33] \frac{a}{3} \; + \; 2\,[19N_f - 153] \frac{a^2}{3} \\
&+ \; \left[- \; 325N_f^2 + 15099N_f - 77139 \right] \frac{a^3}{54} \; + \; O(a^4) \\
\gamma_c(a) \;\; &= \;\; [\alpha - 15] \frac{a}{4} \; + \; \left[- \; 60\alpha^2 - 1020\alpha - 2241 + 200N_f \right] \frac{a^2}{96} \\
&+ \; \left[- \; 22032\alpha^3\zeta_3 + 10908\alpha^3 - 36936\alpha^2\zeta_3 - 161298\alpha^2 - 17928\alpha N_f \right. \\
&\quad + \; 238464\alpha\zeta_3 - 234657\alpha + 11200N_f^2 + 96768N_f\zeta_3 + 206616N_f \\
&\quad \left. + \; 2301696\zeta_3 - 2791386 \right] \frac{a^3}{6912} \; + \; O(a^4) \\
\gamma_{c^i}(a) \;\; &= \;\; - \; 5\,[\alpha + 3] \frac{a}{4} \; + \; \left[- \; 276\alpha^2 - 2028\alpha - 2169 + 200N_f \right] \frac{a^2}{96} \\
&+ \; \left[- \; 59292\alpha^3 + 108864\alpha^2\zeta_3 - 657666\alpha^2 + 81864\alpha N_f + 193104\alpha\zeta_3 \right. \\
&\quad - \; 1137267\alpha + 11200N_f^2 + 96768N_f\zeta_3 + 258456N_f \\
&\quad \left. + \; 1661472\zeta_3 - 2619450 \right] \frac{a^3}{6912} \; + \; O(a^4) \\
\gamma_\psi(a) \;\; &= \;\; \alpha\, a \; + \; \left[- \; 3\alpha^2 + 138\alpha + 262 - 16N_f \right] \frac{a^2}{12} \; + \; O(a^3) \\
&+ \; \left[8532\alpha^3 + 5832\alpha^2\zeta_3 + 71496\alpha^2 - 19224\alpha N_f + 117936\alpha\zeta_3 + 210195\alpha \right. \\
&\quad \left. + \; 1280N_f^2 - 114240N_f + 43848\zeta_3 + 948012 \right] \frac{a^3}{1728} \; + \; O(a^4) \\
\gamma_{\mathcal{O}}(a) \;\; &= \;\; [- \; 15\alpha + 87 - 8N_f] \frac{a}{12} \\
&+ \; \left[- \; 276\alpha^2 - 2028\alpha + 7623 - 1016N_f \right] \frac{a^2}{96} \\
&+ \; \left[- \; 19764\alpha^3 + 36288\alpha^2\zeta_3 - 219222\alpha^2 + 27288\alpha N_f + 64368\alpha\zeta_3 \right. \\
&\quad - \; 379089\alpha + 17600N_f^2 + 32256N_f\zeta_3 - 558072N_f \\
&\quad \left. + \; 553824\zeta_3 + 2418114 \right] \frac{a^3}{2304} \; + \; O(a^4) \; .
\end{aligned} \tag{3.26}
$$

4 Discussion.

We have provided a comprehensive discussion on the three loop $\overline{\mathrm{MS}}$ renormalization of QCD in the maximal abelian gauge. Indeed this article represents the first calculations beyond *one* loop as well as the first for Lie groups other than just $SU(2)$. By explicit computation we have determined all the anomalous dimensions and β-function before deducing the anomalous

dimension of $\mathcal{O}$ at three loops in $\overline{\mbox{MS}}$. Indeed it is the explicit expression for the latter which will be the key to studies of the condensation of $\mathcal{O}$ in the MAG which we hope to examine next. One useful observation from the main results is the relation of the MAG anomalous dimensions to those of other gauges and in particular the Curci-Ferrari gauge. That the results for the latter appear in the formal limit $N^d_A/N^o_A \rightarrow 0$ is reassuring, though their prior existence was also of a more practical use in helping to establish the veracity of the final MAG renormalization group functions. Though from the actual structure of the final expressions it is clear that they could not be constructed from knowledge of the same anomalous dimensions in the Landau or Curci-Ferrari gauges.

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A Feynman rules.

In this appendix we record the Feynman rules we used for the maximal abelian gauge fixing in momentum space which are derived from (2.5) and (2.13) using a symbolic manipulation programme written in FORM. For the propagators we have

$$
\begin{aligned}
\langle A^a_\mu(p) A^b_\nu(-p) \rangle &= - \frac{\delta^{ab}}{p^2} \left[\eta_{\mu\nu} ~-~ (1-\alpha) \frac{p_\mu p_\nu}{p^2} \right] \\
\langle A^i_\mu(p) A^j_\nu(-p) \rangle &= - \frac{\delta^{ij}}{p^2} \left[\eta_{\mu\nu} ~-~ (1-\bar{\alpha}) \frac{p_\mu p_\nu}{p^2} \right] \\
\langle c^a(p) \bar{c}^b(-p) \rangle &= \frac{\delta^{ab}}{p^2} \\
\langle c^i(p) \bar{c}^j(-p) \rangle &= \frac{\delta^{ij}}{p^2} \\
\langle \psi(p) \bar{\psi}(-p) \rangle &= \frac{\not{p}}{p^2}
\end{aligned}
\tag{A.1}
$$

where p is the momentum. The non-zero 3- and 4-point vertices are

$$
\begin{aligned}
\langle A^a_\mu(p_1) \bar{\psi}(p_2) \psi(p_3) \rangle &= g T^a \gamma_\mu \\
\langle A^i_\mu(p_1) \bar{\psi}(p_2) \psi(p_3) \rangle &= g T^i \gamma_\mu \\
\langle A^a_\mu(p_1) \bar{c}^b(p_2) c^c(p_3) \rangle &= - ~ i g f^{abc} \left(- \frac{1}{2} p_1 - p_3 \right)_\mu \\
\langle A^a_\mu(p_1) \bar{c}^b(p_2) c^k(p_3) \rangle &= - ~ i g f^{abk} \left(- \zeta p_3 \right)_\mu \\
\langle A^a_\mu(p_1) \bar{c}^j(p_2) c^c(p_3) \rangle &= - ~ i g f^{acj} \left(p_1 + p_3 \right)_\mu \\
\langle A^i_\mu(p_1) \bar{c}^b(p_2) c^c(p_3) \rangle &= - ~ i g f^{bci} \left(- p_1 - 2 p_3 + p_3 \zeta \right)_\mu \\
\langle A^a_\mu(p_1) A^b_\nu(p_2) A^c_\sigma(p_3) \rangle &= i g f^{abc} \left(\eta_{\nu\sigma} (p_2 - p_3)_\mu + \eta_{\sigma\mu} (p_3 - p_1)_\nu + \eta_{\mu\nu} (p_1 - p_2)_\sigma \right) \\
\langle A^a_\mu(p_1) A^b_\nu(p_2) A^c_\sigma(p_3) A^d_\rho(p_4) \rangle &= - ~ \Big[f^{abcd}_d \left(- \eta_{\mu\sigma} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\sigma} \right) + f^{acbd}_d \left(- \eta_{\mu\nu} \eta_{\sigma\rho} + \eta_{\mu\rho} \eta_{\nu\sigma} \right) \\
&\quad + f^{adbc}_d \left(- \eta_{\mu\nu} \eta_{\sigma\rho} + \eta_{\mu\sigma} \eta_{\nu\rho} \right) + f^{abcd}_o \left(- \eta_{\mu\sigma} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\sigma} \right) \\
&\quad + f^{acbd}_o \left(- \eta_{\mu\nu} \eta_{\sigma\rho} + \eta_{\mu\rho} \eta_{\nu\sigma} \right) + f^{adbc}_o \left(- \eta_{\mu\nu} \eta_{\sigma\rho} + \eta_{\mu\sigma} \eta_{\nu\rho} \right) \Big]
\end{aligned}
$$

$$
\begin{aligned}
\langle A^a_\mu(p_1)A^b_\nu(p_2)A^c_\sigma(p_3)A^l_\rho(p_4)\rangle &= -\,g\Big(f^{abcl}_o\left(-\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}\right)\\
&\quad +f^{acbl}_o\left(-\eta_{\mu\nu}\eta_{\sigma\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}\right)\\
&\quad +f^{albc}_o\left(-\eta_{\mu\nu}\eta_{\sigma\rho}+\eta_{\mu\sigma}\eta_{\nu\rho}\right)\Big)\\
\langle A^a_\mu(p_1)A^b_\nu(p_2)A^k_\sigma(p_3)A^l_\rho(p_4)\rangle &= -\,g\Big(f^{akbl}_o\Big(-\eta_{\mu\nu}\eta_{\sigma\rho}+\frac{\zeta(2-\zeta)}{2\alpha}\eta_{\mu\sigma}\eta_{\nu\rho}-\frac{1}{2\alpha}\eta_{\mu\sigma}\eta_{\nu\rho}\\
&\qquad +\eta_{\mu\rho}\eta_{\nu\sigma}\Big)\\
&\quad +f^{albk}_o\Big(-\eta_{\mu\nu}\eta_{\sigma\rho}+\eta_{\mu\sigma}\eta_{\nu\rho}+\frac{\zeta(2-\zeta)}{2\alpha}\eta_{\mu\rho}\eta_{\nu\sigma}\\
&\qquad -\frac{1}{2\alpha}\eta_{\mu\rho}\eta_{\nu\sigma}\Big)\\
&\quad +f^{bkal}_o\left(\frac{\zeta(2-\zeta)}{2\alpha}\eta_{\mu\rho}\eta_{\nu\sigma}-\frac{1}{2\alpha}\eta_{\mu\rho}\eta_{\nu\sigma}\right)\\
&\quad +f^{blak}_o\left(\frac{\zeta(2-\zeta)}{2\alpha}\eta_{\mu\sigma}\eta_{\nu\rho}-\frac{1}{2\alpha}\eta_{\mu\sigma}\eta_{\nu\rho}\right)\Big)\\
\langle A^a_\mu(p_1)A^b_\nu(p_2)\bar{c}^c(p_3)c^d(p_4)\rangle &= -\,g\left(f^{acbd}_d\left(-\eta_{\mu\nu}+\zeta\eta_{\mu\nu}\right)+f^{bcad}_d\left(-\eta_{\mu\nu}+\zeta\eta_{\mu\nu}\right)\right)\\
\langle A^a_\mu(p_1)A^j_\nu(p_2)\bar{c}^c(p_3)c^d(p_4)\rangle &= -\,g\left(f^{adcj}_o\left(-\eta_{\mu\nu}+\zeta\eta_{\mu\nu}\right)+f^{ajcd}_o\left(\frac{1}{2}\eta_{\mu\nu}-\frac{\zeta}{2}\eta_{\mu\nu}\zeta\right)\right)\\
\langle A^a_\mu(p_1)A^j_\nu(p_2)\bar{c}^c(p_3)c^l(p_4)\rangle &= -\,g\left(f^{ajcl}_o\left(\eta_{\mu\nu}-\zeta\eta_{\mu\nu}\right)+f^{alcj}_o\left(-\zeta\eta_{\mu\nu}+\eta_{\mu\nu}\right)\right)\\
\langle A^i_\mu(p_1)A^j_\nu(p_2)\bar{c}^c(p_3)c^d(p_4)\rangle &= -\,g\left(f^{cidj}_o\left(\eta_{\mu\nu}-\zeta\eta_{\mu\nu}\right)+f^{cjdi}_o\left(\zeta\eta_{\mu\nu}-\eta_{\mu\nu}\right)\right)\\
\langle\bar{c}^a(p_1)c^b(p_2)\bar{c}^c(p_3)c^d(p_4)\rangle &= -\,g\left(\alpha f^{acbd}_d-\frac{\alpha}{4}f^{abcd}_o+\frac{\alpha}{2}f^{acbd}_o-\frac{\alpha}{4}f^{adbc}_o\right)\\
\langle\bar{c}^a(p_1)c^b(p_2)\bar{c}^c(p_3)c^l(p_4)\rangle &= -\,g\left(-\frac{\alpha}{2}f^{abcl}_o+\frac{\alpha}{2}f^{acbl}_o-\frac{\alpha}{2}f^{albc}_o\right)\\
\langle\bar{c}^a(p_1)c^j(p_2)\bar{c}^c(p_3)c^l(p_4)\rangle &= -\,g\left(-\alpha f^{ajcl}_o+\alpha f^{alcj}_o\right)
\end{aligned}
\qquad (A.2)
$$

where the momentum flow for each field is into the vertex. We note that we have recorded the Feynman rules as generated from the full MAG Lagrangian, using a Form routine, without recourse to the simplifying properties of the Jacobi identity of the Lie algebra. For example, the final rule of the non-zero set, (A.2), actually vanishes after application of a Jacobi identity. However, in the construction of the routines to perform the overall calculation, we have relegated all the algebra associated with the group theory to a common Form module which encodes the necessary simplifying lemmas. That module is placed after the module where the above Feynman rules are substituted. The remaining Feynman rules for the 3- and 4-point vertices which have not been recorded above are trivially zero due to the fact that they would involve either two or more centre indices in the case of the 3-point vertices or three or more centre indices for the 4-point vertices.

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